

Electromagnetic interaction between two uniformly moving charged particles: a geometrical derivation using Minkowski diagrams

Călin Galeriu

Physics Department, Clark University, Worcester, MA, 01610, USA

Abstract

This paper presents an intuitive, geometrical derivation of the relativistic addition of velocities, and of the electromagnetic interaction between two uniformly moving charged particles, based on 2 spatial + 1 temporal dimensional Minkowski diagrams. We calculate the relativistic addition of velocities by projecting the world-line of the particle on the spatio-temporal planes of the reference frames considered. We calculate the real component of the electromagnetic 4-force, in the proper reference frame of the source particle, starting from the Coulomb force generated by a charged particle at rest. We then obtain the imaginary component of the 4-force, in the same reference frame, from the requirement that the 4-force be orthogonal to the 4-velocity. The 4-force is then projected on a real 3 dimensional space to get the Lorentz force.

I. INTRODUCTION

Special Relativity, as presented in today's textbooks, is a complex mathematical theory. The 1 spatial + 1 temporal dimensional Minkowski diagrams, which initially introduce the Lorentz transformation, the time dilatation and the length contraction, are soon put aside in favor of an approach based on differential calculus and linear algebra. One gets

little intuitive understanding of the law of relativistic addition of velocities, and of the fact that "magnetism is a kind of 'second-order' effect arising from relativistic changes in the electric fields of moving charges" [1]. However, by introducing only slightly more elaborate Minkowski diagrams, and using geometrical derivations, one can get back the intuitive understanding, to the great delight of the physicist who still believes in the spirit of Descartes' philosophy.

II. REAL PLANE AND COMPLEX PLANE: SAME TRIGONOMETRY

The complex plane, like the real plane, is a two-dimensional (2D) vector space. The scalar products for the real plane (1) and for the complex plane (2) are defined as follows

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1 \quad \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = 1 \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = 0 \quad (1)$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1 \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = -1 \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{x}} = 0 \quad (2)$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and respectively $\hat{\mathbf{x}}, \hat{\mathbf{i}}$ are the basis vectors.

Since both planes have a scalar product, one can talk about orthogonal vectors (their scalar product is zero) and about the magnitude of a vector (the square root of the scalar product of a vector with itself). This allows us to define the circle (the geometric locus of the points equally spaced from a given point), the angle in radians (the length, between two points of a circle of radius one, measured along the circumference), and the trigonometric functions sine and cosine (the magnitudes of the projections of a radius one vector on the two coordinate axes). From these definitions it follows that, for both the real and the complex planes, one has the relations:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \quad (3)$$

$$\left[\frac{d}{d\alpha} \sin(\alpha)\right]^2 + \left[\frac{d}{d\alpha} \cos(\alpha)\right]^2 = 1. \quad (4)$$

From (3)-(4) one can get the derivatives of the trigonometric functions, the Taylor series expansions of sine and cosine, and then all the well known trigonometric relations. The only detail we have to keep in mind is that the angle α in the real plane is a real number, while in the complex plane it is a purely imaginary number, due to the non-positive definite scalar product used in the last case. There are a few more relevant differences, which we can best point out if we represent the complex plane as an Euclidean plane. Two vectors in the complex plane are orthogonal if they make the same angle with the bisecting line of the first quadrant. The circle in the complex plane looks like a hyperbola [2]. Not any line passing through the origin intersects the right (or left) branch of the hyperbola. This means that there are pairs of lines passing through the origin to which we cannot assign an angle. However, for the triangles we will be working with, the ratio of segments behaves as if it were an angle, of negative value. The true angle is obtained by symmetry with respect to the first bisecting line, as the pair α and $-\alpha$ indicates in Figure 1.

III. RELATIVISTIC ADDITION OF VELOCITIES

Consider a reference frame K' which is moving with a velocity $\mathbf{V} = V\hat{\mathbf{x}}$ relative to another one K , and a particle moving with a velocity $\mathbf{v}' = v'_x\hat{\mathbf{x}}' + v'_y\hat{\mathbf{y}}' + v'_z\hat{\mathbf{z}}'$ in the reference frame K' . The reference frames are chosen such that their origins and the particle coincide at the space-time point O , as shown in Figure 1. Notice that $\hat{\mathbf{y}} = \hat{\mathbf{y}}'$ and $\hat{\mathbf{z}} = \hat{\mathbf{z}}'$, because \mathbf{V} has a component only in the x direction. The Oz axis is not plotted, but is similar to the Oy axis. The question is: What is the velocity $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ of the particle in the reference frame K ?

The world-line OP of the particle is projected on the complex planes (x,O,ict) , (y,O,ict) , (z,O,ict) , (x',O,ict') , (y',O,ict') , (z',O,ict') , and the resulting angles from the respective projections give the components of the velocity of the particle in the two reference frames considered. For the situation considered the planes (x,O,ict) and (x',O,ict') coincide. It is seen from Figure 1 that

$$\tan(-\alpha) = \frac{EF}{OE} = \frac{V}{ic} \quad \tan(\alpha) = -\frac{V}{ic} \quad (5)$$

$$\tan(-\beta) = \frac{DC}{OD} = \frac{v'_x}{ic} \quad \tan(\beta) = -\frac{v'_x}{ic} \quad (6)$$

$$\tan(-\gamma) = \frac{DA}{OD} = \frac{v'_y}{ic} \quad \tan(\gamma) = -\frac{v'_y}{ic} \quad (7)$$

$$\tan(-\delta) = \frac{EB}{OE} = \frac{v_y}{ic} \quad \tan(\delta) = -\frac{v_y}{ic} \quad (8)$$

$$\tan(-\theta) = \frac{EC}{OE} = \frac{v_x}{ic} \quad \tan(\theta) = -\frac{v_x}{ic}. \quad (9)$$

In order to express v_x and v_y as functions of V, v'_x and v'_y we need to express δ and θ as functions of α, β and γ .

In the plane (x,O,ict) of the Lorentz boost the addition of velocities is based on the addition of angles [3]

$$\theta = \alpha + \beta \quad (10)$$

$$\tan(\theta) = \tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \tan(\beta)}. \quad (11)$$

From (11), by substitution of the tangents (5)-(9), it follows that

$$v_x = \frac{V + v'_x}{1 + V v'_x / c^2}. \quad (12)$$

Two rectangles, APCD and BPCE, result from the projection process. It is evident that

$$\frac{CP}{OC} = \frac{EB}{OC} = \frac{EB}{OE} \frac{OE}{OC} = \tan(-\delta) \cos(-\theta) \quad (13)$$

$$\frac{CP}{OC} = \frac{DA}{OC} = \frac{DA}{OD} \frac{OD}{OC} = \tan(-\gamma) \cos(-\beta). \quad (14)$$

From (13)-(14) it follows that

$$\tan(\delta) = \frac{\cos(\beta) \tan(\gamma)}{\cos(\alpha + \beta)} = \frac{\tan(\gamma)}{\cos(\alpha) [1 - \tan(\alpha) \tan(\beta)]}. \quad (15)$$

By substitutions of the tangents (5)-(8) and of $\cos(\alpha) = [1 + \tan^2(\alpha)]^{-1/2}$ we get

$$v_y = \frac{v'_y (1 - V^2 / c^2)^{1/2}}{1 + V v'_x / c^2}. \quad (16)$$

A similar expression is obtained for the v_z component.

IV. ELECTROMAGNETIC INTERACTION BETWEEN TWO UNIFORMLY MOVING CHARGED PARTICLES

Consider two charged particles (with charges Q_1 and Q_2) at some arbitrary positions, moving with arbitrary, but uniform, velocities. We orient our 3D reference frame in such a way that the first particle (which generates the field) is initially at the origin, moving along the Ox axis with velocity $\mathbf{V} = V\hat{\mathbf{x}}$, and the vector $\mathbf{R} = R\cos(\theta)\hat{\mathbf{x}} + R\sin(\theta)\hat{\mathbf{y}}$ connecting the two particles is in the (x,O,y) plane. The angle between \mathbf{R} and the Ox axis is θ . The second particle (subject to the electromagnetic field generated by the first one) is moving with velocity $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$. A section through the (x,O,y) plane can be seen in Figure 2. The first particle is at point O and the second one is at point A.

A. Analytical calculation of the Lorentz force

The electric field (in Gaussian units) generated by the first particle at the position of the second particle is [4,5]

$$\mathbf{E} = \frac{Q_1\mathbf{R}}{R^3}\left(1 - \frac{V^2}{c^2}\right)\left[1 - \frac{V^2}{c^2}\sin^2(\theta)\right]^{-3/2}. \quad (17)$$

The magnetic field generated by the first particle is

$$\mathbf{H} = \frac{1}{c}\mathbf{V} \times \mathbf{E}. \quad (18)$$

The Lorentz force acting on the second particle is

$$\mathbf{F} = Q_2\mathbf{E} + \frac{Q_2}{c}\mathbf{v} \times \mathbf{H}. \quad (19)$$

From (17)-(19) the Cartesian components of the force [6] are obtained

$$F_x = \frac{Q_1Q_2}{R^2}\left(1 - \frac{V^2}{c^2}\right)\left[1 - \frac{V^2}{c^2}\sin^2(\theta)\right]^{-3/2}[\cos(\theta) + \sin(\theta)\frac{v_yV}{c^2}] \quad (20)$$

$$F_y = \frac{Q_1Q_2}{R^2}\left(1 - \frac{V^2}{c^2}\right)\left[1 - \frac{V^2}{c^2}\sin^2(\theta)\right]^{-3/2}\sin(\theta)\left(1 - \frac{v_xV}{c^2}\right) \quad (21)$$

$$F_z = 0. \quad (22)$$

B. Geometrical derivation of the Lorentz force

The force components (20)-(22) can be obtained in a more graphical way, if we start with the Coulomb force generated by a charged particle at rest. One key assumption or experimental fact is that in a frame where all the source charges producing an electric field \mathbf{E} are at rest, the force on a charge q is given by $\mathbf{F} = q\mathbf{E}$ independent of the velocity of the charge in that frame [7]. The reference frame K' in which the source particle is at rest is moving with velocity \mathbf{V} relative to the original frame K .

In the reference frame K the particle at A is observed to interact with the particle at O . The distance between particles is R , the length of the segment OA .

In the reference frame K' the particle at A is observed to interact with the particle at B , where the segment BA is a position vector \mathbf{R}' parallel to the plane (x', O, y') . The following construction gives the position of point B : the segment AE is parallel to Oy and intersects the Ox axis at E , whereas the segment EB is parallel to Ox' and intersects the world-line CO at B . BD projects the point B on the Ox axis at D .

Relative to K' , the particle at B exerts a radial Coulomb force on the particle at A . This force (in Gaussian units) is

$$\mathbf{F}' = \frac{Q_1 Q_2}{R'^3} \mathbf{R}' \quad (23)$$

where $\mathbf{R}' = R'[\cos(\theta')\hat{\mathbf{x}}' + \sin(\theta')\hat{\mathbf{y}}']$.

The key point in getting the force \mathbf{F} in the reference frame K is to notice that the force, in any reference frame considered, is given by the projection on the real 3D space of that frame of the 4-force \mathcal{F} (which is a Minkowski-space vector), that is

$$\mathcal{F} = \mathcal{F}_{\text{real}} + \mathcal{F}_{\text{imag}} = \gamma(v)\mathbf{F} + \hat{\mathbf{i}}\gamma(v)\frac{P}{c} \quad (24)$$

$$\mathcal{F} = \mathcal{F}'_{\text{real}} + \mathcal{F}'_{\text{imag}} = \gamma(v')\mathbf{F}' + \hat{\mathbf{i}}'\gamma(v')\frac{P'}{c} \quad (25)$$

where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$ and $P = \mathbf{F} \cdot \mathbf{v}$.

We will obtain the 4-force \mathcal{F} from its real and imaginary components ($\mathcal{F}'_{\text{real}}$ and $\mathcal{F}'_{\text{imag}}$) in the reference frame K' , then we will decompose the same 4-force into its real and imaginary components ($\mathcal{F}_{\text{real}}$ and $\mathcal{F}_{\text{imag}}$) in the reference frame K . The Lorentz force we are looking for is just $\mathbf{F} = \mathcal{F}_{\text{real}}/\gamma(\mathbf{v})$.

From (23)-(25) it follows that

$$\mathcal{F}'_{\text{real}} = \gamma(v') \frac{Q_1 Q_2}{R'^3} \mathbf{R}'. \quad (26)$$

To get the imaginary component $\mathcal{F}'_{\text{imag}}$ we use the orthogonality between the 4-force and the 4-velocity, $\mathcal{F} \cdot \mathbf{V} = 0$, where the 4-velocity is $\mathbf{V} = \gamma(v')\mathbf{v}' + \hat{\mathbf{i}}'\gamma(v')c$. The orthogonality condition leads to

$$\gamma^2(v') \frac{Q_1 Q_2}{R'^2} \frac{\mathbf{R}' \cdot \mathbf{v}'}{R'} + \mathcal{F}'_{\text{imag}} \cdot \hat{\mathbf{i}}'\gamma(v')c = 0 \quad (27)$$

$$\mathcal{F}'_{\text{imag}} = \hat{\mathbf{i}}'\gamma(v') \frac{Q_1 Q_2}{R'^2} \frac{v'_{\text{rad}}}{c} \quad (28)$$

where the radial component of the velocity is

$$v'_{\text{rad}} = \frac{\mathbf{R}' \cdot \mathbf{v}'}{R'} = v'_x \cos(\theta') + v'_y \sin(\theta'). \quad (29)$$

The components of the force \mathbf{F} in the reference frame K are given by the projection of the 4-force \mathcal{F} on the 3D real space of K . An easy way to do this is to notice that we can decompose $\mathcal{F}'_{\text{real}}$ (which has the direction of the segment BA) and $\mathcal{F}'_{\text{imag}}$ (which has the direction of the segment BO) into sums of 4-vectors, each of the 4-vectors being parallel to one of the axes of the reference frame K :

$$\mathbf{r}_{BA} = \mathbf{r}_{BD} + \mathbf{r}_{DE} + \mathbf{r}_{EA} \quad (30)$$

$$\mathbf{r}_{BO} = \mathbf{r}_{BD} + \mathbf{r}_{DO} \quad (31)$$

Because these expansions do not involve any component along the Oz axis, this simply means that $F_z = 0$. The projections of the 4-force on the Ox and Oy axes are

$$\gamma(v)F_x = \mathcal{F}'_{\text{real}} \frac{DE}{BA} + \mathcal{F}'_{\text{imag}} \frac{DO}{BO} \quad (32)$$

$$\gamma(v)F_y = \mathcal{F}'_{\text{real}} \frac{EA}{BA}. \quad (33)$$

The lengths of the various segments needed above are as follows:

$$AO = R \quad (34)$$

$$EA = AO \sin(\theta) = R \sin(\theta) \quad (35)$$

$$OE = AO \cos(\theta) = R \cos(\theta) \quad (36)$$

$$BE = OE \cos(\alpha) = R \cos(\theta) \cos(\alpha) \quad (37)$$

$$DE = BE \cos(\alpha) = R \cos(\theta) \cos^2(\alpha) \quad (38)$$

$$AB = (AE^2 + BE^2)^{1/2} = R \cos(\alpha) [1 + \tan^2(\alpha) \sin^2(\theta)]^{1/2} = R' \quad (39)$$

We also notice that $DO/BO = \sin(-\alpha)$. The force components in (32)-(33) become

$$F_x = \frac{\gamma(v')}{\gamma(v)} \frac{Q_1 Q_2}{R^2} \frac{\cos(\theta)}{\cos(\alpha) [1 + \tan^2(\alpha) \sin^2(\theta)]^{3/2}} + i \frac{\gamma(v')}{\gamma(v)} \frac{Q_1 Q_2}{R^2} \frac{v'_{\text{rad}}}{c} \frac{\sin(-\alpha)}{\cos^2(\alpha) [1 + \tan^2(\alpha) \sin^2(\theta)]} \quad (40)$$

$$F_y = \frac{\gamma(v')}{\gamma(v)} \frac{Q_1 Q_2}{R^2} \frac{\sin(\theta)}{\cos^3(\alpha) [1 + \tan^2(\alpha) \sin^2(\theta)]^{3/2}}. \quad (41)$$

We can also calculate

$$\sin(\theta') = \frac{EA}{AB} = \frac{\sin(\theta)}{\cos(\alpha) [1 + \tan^2(\alpha) \sin^2(\theta)]^{1/2}} \quad (42)$$

$$\cos(\theta') = \frac{BE}{AB} = \frac{\cos(\theta)}{[1 + \tan^2(\alpha) \sin^2(\theta)]^{1/2}}. \quad (43)$$

If the velocity of the particle at A has the components v_x, v_y, v_z , as measured in the reference frame K, and K is moving with the velocity $\mathbf{V}' = -V\hat{\mathbf{x}}'$ relative to K', then the particle will have the following components of the velocity (compare with equations (12) and (16)) in the reference frame K'

$$v'_x = \frac{v_x - V}{1 - Vv_x/c^2} \quad v'_y = \frac{v_y(1 - V^2/c^2)^{1/2}}{1 - Vv_x/c^2} \quad v'_z = \frac{v_z(1 - V^2/c^2)^{1/2}}{1 - Vv_x/c^2}. \quad (44)$$

With these components we find that

$$\gamma(v') = \gamma(v) \frac{(1 - Vv_x/c^2)}{(1 - V^2/c^2)^{1/2}} \quad (45)$$

and the radial velocity (29) becomes

$$v'_{\text{rad}} = \frac{(v_x - V) \cos(\alpha) \cos(\theta) + v_y(1 - V^2/c^2)^{1/2} \sin(\theta)}{(1 - Vv_x/c^2) \cos(\alpha) [1 + \tan^2(\alpha) \sin^2(\theta)]^{1/2}}. \quad (46)$$

Substituting $\gamma(v')$ and v'_{rad} in (40)-(41), and also using the fact that $\sin(\alpha) = i(V/c)\gamma(V)$, $\cos(\alpha) = \gamma(V)$ and $\tan(\alpha) = iV/c$, we finally obtain the components in (20)-(21).

V. CONCLUSIONS

We have presented a geometrical calculation of the relativistic addition of velocities, and of the electromagnetic interaction between two uniformly moving charged particles. The geometrical approach used is an elegant, more intuitive and alternative way of obtaining these important results of Special Relativity. We hope our work will usefully complement other pedagogical efforts [8–10] centered on Minkowski space diagrams.

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FIGURES

FIG. 1. Relativistic addition of velocities. The world-line OP is projected on various spatio-temporal planes. OA is the projection on (y', O, ict') , OB is the projection on (y, O, ict) and OC is the projection on (x, O, ict) . The planes (x, O, ict) and (x', O, ict') coincide.

FIG. 2. Electromagnetic interaction between two uniformly moving charged particles. CO is the world-line of the source particle, and AG is the world-line of the test particle. In the proper reference frame of the source particle there is a Coulomb force directed along the BA radial direction.

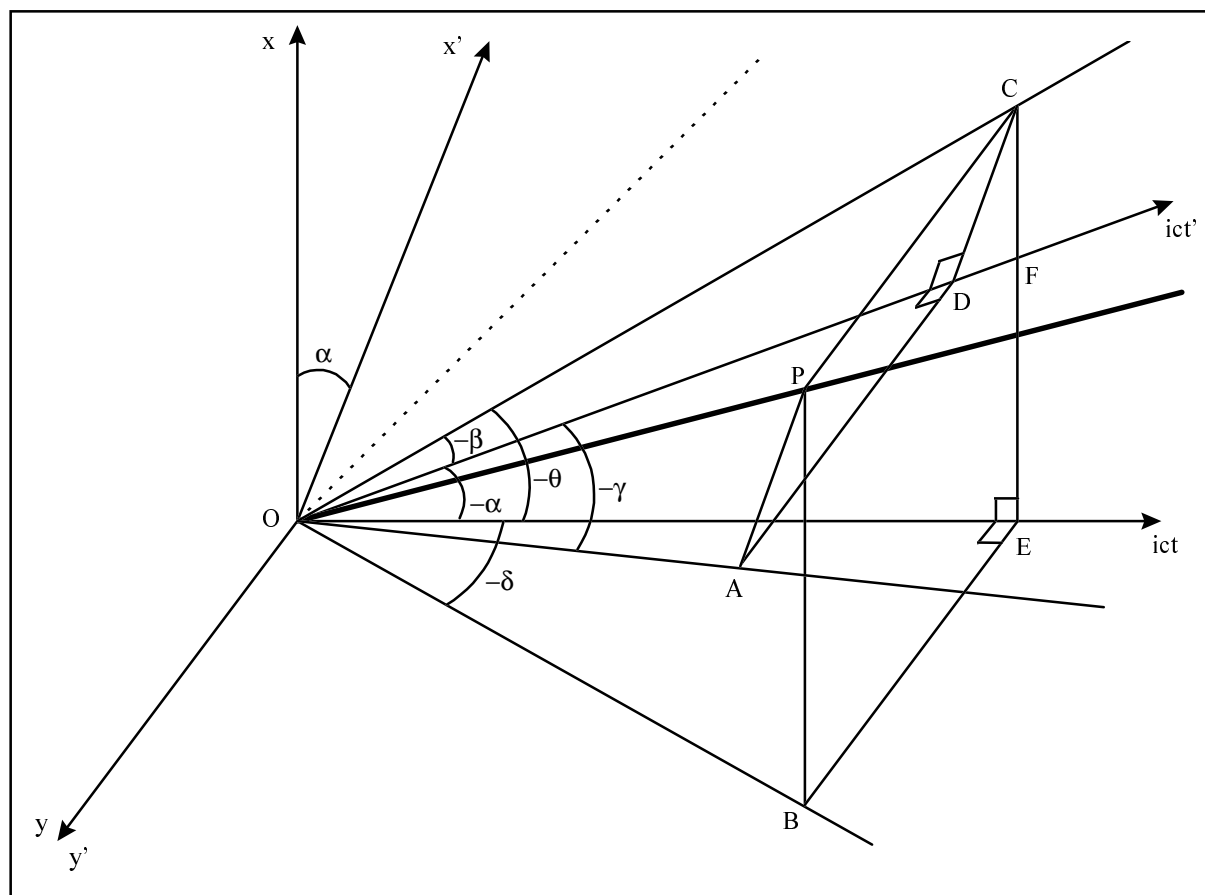


Figure 1, Calin Galeriu, Electromagnetic interaction

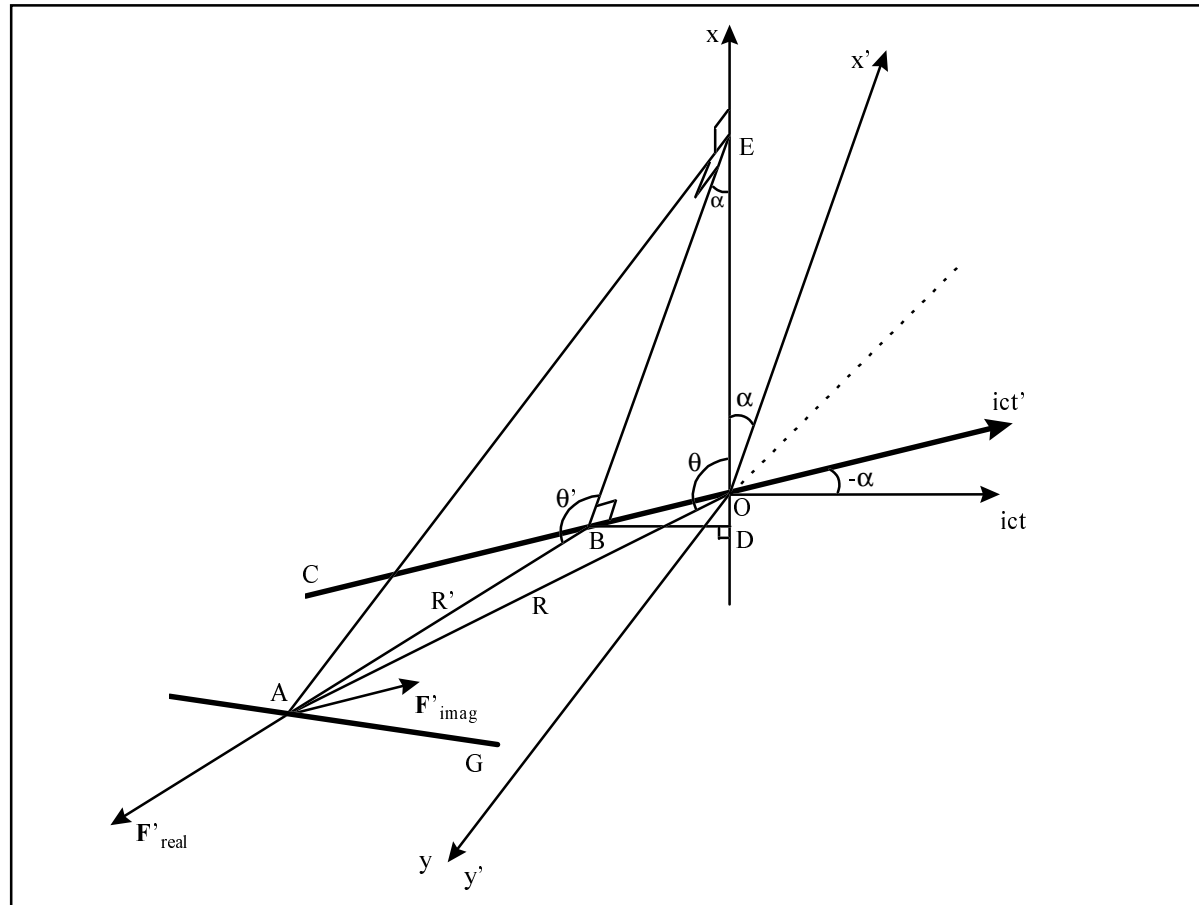


Figure 2, Calin Galeriu, Electromagnetic interaction